Investigating Maximal Lifespan on the Vietoris-Rips Complex

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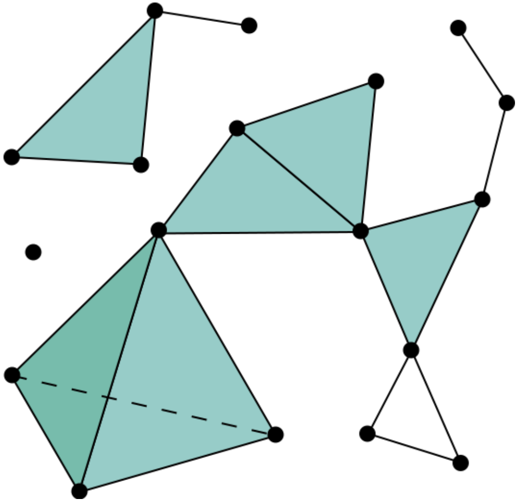
Constructing Simplicial Complexes

A Geometric Approach:
A set of Interconnected *Simplices*:

- 0-simplex: Points
- 1-simplex: Line Segments
- 2-simplex: Triangles
- 3-simplex: Tetrahedrons
- 4-simplex: Pentachorons

  $\dots$
- k-simplex: $[v_0, v_1, \dots, v_k]$



$$X = \{v_0, v_1, v_2, \ldots, v_k\} \subset \mathbb{R}^n$$

The set X defines a k-simplex: $\sigma = [v_0, v_1, \ldots, v_k]$, i.e. a simplex of dimension k.

$$\begin{aligned} \sigma &= [v_0, v_1, \ldots, v_k] \\ &= \left\{ \sum_{i=0}^{k} t_i v_i \,\middle|\, t_i \geq 0, \sum t_i = 1 \right\} \end{aligned}$$

We will define a simplicial complex S on a finite set of vertices X to be a set of simplices generated by subsets of X. Moreover, for every simplex $\sigma \in S$, every face of σ must also be in S.

Brief Overview of 1-Cycles

Often what will appear in simplicial complexes are holes or gaps in the graph.

A 2-dimensional hole in a graph, like the hole of a donut or the handle of a coffee mug is called a 1-Cycle. It is not the same as a 2-Cycle which is easiest though of as a completely walled off empty space, like the interior of hollow sphere. Today, we will be considering only 1-cycles.

The Vietoris-Rips Complex

Let $X = \{v_0, v_1, \ldots, v_k\} \subset \mathbb{R}^n$ be a set of vertices. We can define the *Vietoris-Rips Complex* at value $\epsilon \geq 0$; $V_\epsilon(X)$, to be the simplicial complex defined on the set X given the following condition:

$$\forall v_a, v_b \in \{v_{i_0}, v_{i_1}, \ldots, v_{i_l}\} \subset X, \quad d(v_a, v_b) \leq \epsilon$$

$$\Rightarrow [v_{i_0}, v_{i_1}, \ldots, v_{i_l}] \in V_\epsilon(X).$$

Conditions

Consider a finite set of vertices $X = \{v_0, v_1, v_2, \ldots, v_k\} \subset \mathbb{R}^n$ displaying the following characteristics:

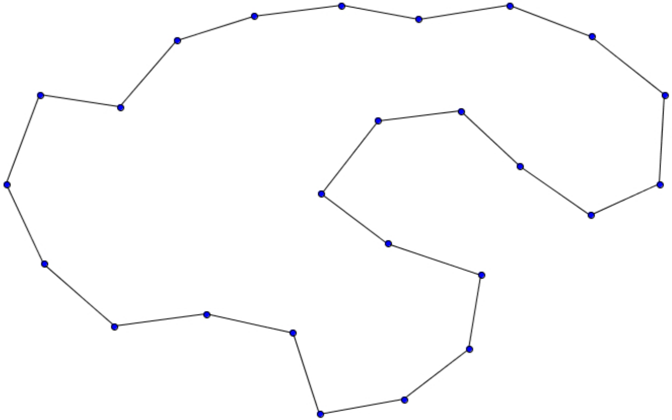
- The Vietoris-Rips Complex begins displaying a 1-cycle at $\epsilon = \alpha$ and not before:

$$\beta_1(V_\epsilon(X)) = 0 \text{ for } \epsilon < \alpha, \quad \beta_1(V_\alpha(X)) = 1.$$

- For every $v_i \in X$, there exist precisely two distinct elements $v_j, v_k \in X, v_j, v_k \neq v_i$ such that $d(v_i, v_j), d(v_i, v_k) \leq \alpha$.
- At $\epsilon = \alpha$ there exists only one component:

$$\beta_0(V_\alpha(X)) = 1.$$

Note: these three conditions necessitate that the $V_\alpha(X)$ is simply a loop of 1-simplices that is homeomorphic to S^1. Furthermore, we have enumerated the indices s.t. v_i's two neighbors will be v_{i-1} and v_{i+1}.



Definitions

Define α as the *Birth*-value of the 1-cycle.

Define the *Death*-value of the 1-cycle as:

$$\gamma = \min(\epsilon > \alpha : \beta_1(V_\epsilon(X)) = 0).$$

The death-value should be interpreted as the smallest value of ϵ when the 1-cycle born at α is filled in with 2-simplices.

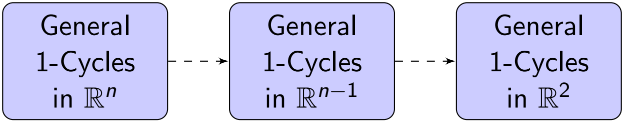
Define the *Lifespan* of the 1-cycle to be $l = \gamma - \alpha$.

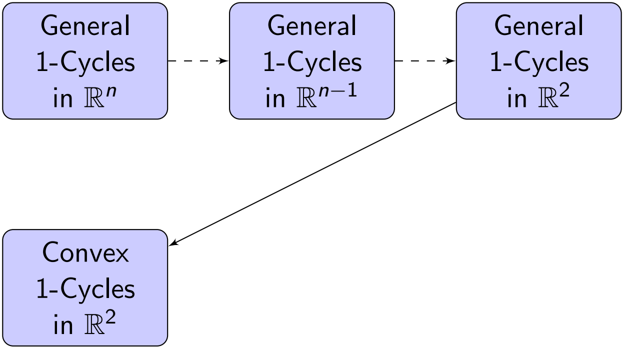
Primary Question and Hypothesis

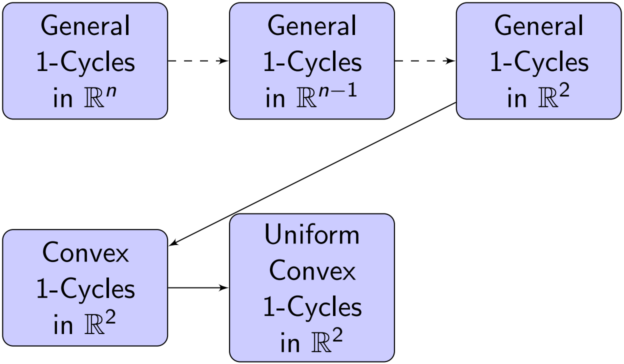
- It is our goal to try and discover the maximal lifespan $\Leftrightarrow$ death-value of a general 1-cycle given a birth value α and the number of vertices $k + 1$ under the rips complex as ϵ increases.
- Our standing hypothesis is that maximal lifespan is achieved with taking a subset of $k + 1$ vertices from $\mathbb{S}^1$ where each point is equally spaced at distance α away from its neighbors.

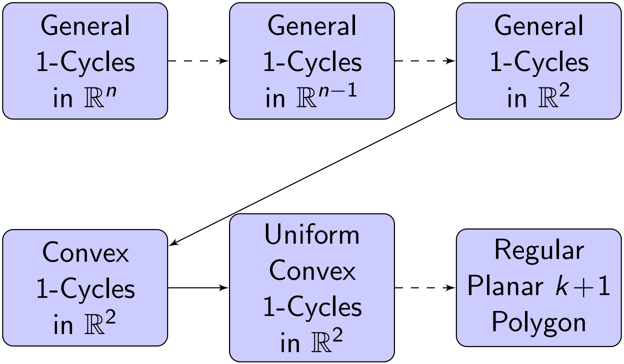
General
1-Cycles
in $\mathbb{R}^n$

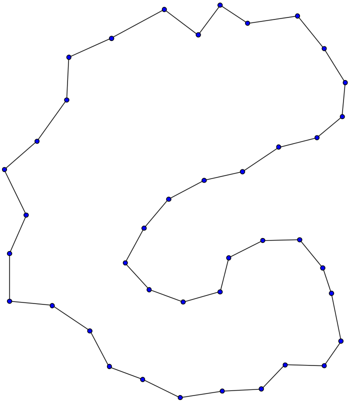
$$\begin{tikzcd}
\boxed{\begin{array}{c}\text{General}\\ \text{1-Cycles}\\ \text{in } \mathbb{R}^n\end{array}} \arrow[r, dashed] & \boxed{\begin{array}{c}\text{General}\\ \text{1-Cycles}\\ \text{in } \mathbb{R}^{n-1}\end{array}}
\end{tikzcd}$$

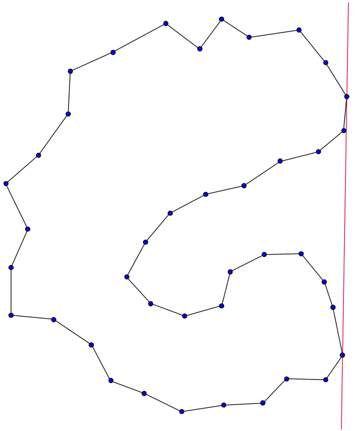

General 1-Cycles in $\mathbb{R}^n$
General 1-Cycles in $\mathbb{R}^{n-1}$
General 1-Cycles in $\mathbb{R}^2$

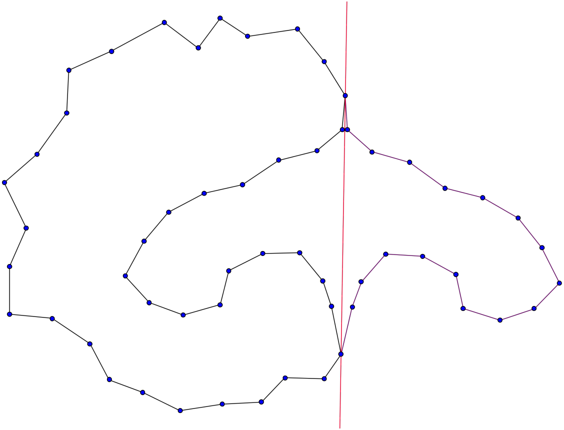

General 1-Cycles in $\mathbb{R}^n$
General 1-Cycles in $\mathbb{R}^{n-1}$
General 1-Cycles in $\mathbb{R}^2$
Convex 1-Cycles in $\mathbb{R}^2$

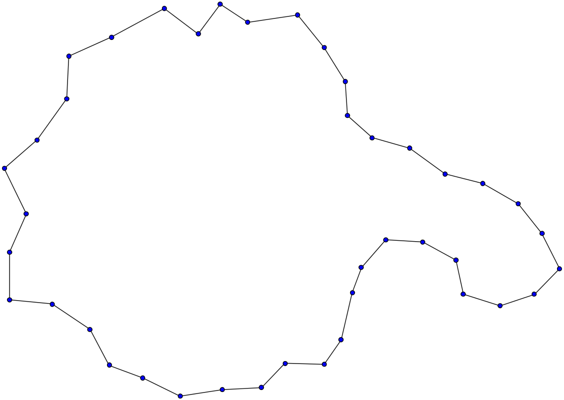

General 1-Cycles in $\mathbb{R}^n$
General 1-Cycles in $\mathbb{R}^{n-1}$
General 1-Cycles in $\mathbb{R}^2$
Convex 1-Cycles in $\mathbb{R}^2$
Uniform Convex 1-Cycles in $\mathbb{R}^2$

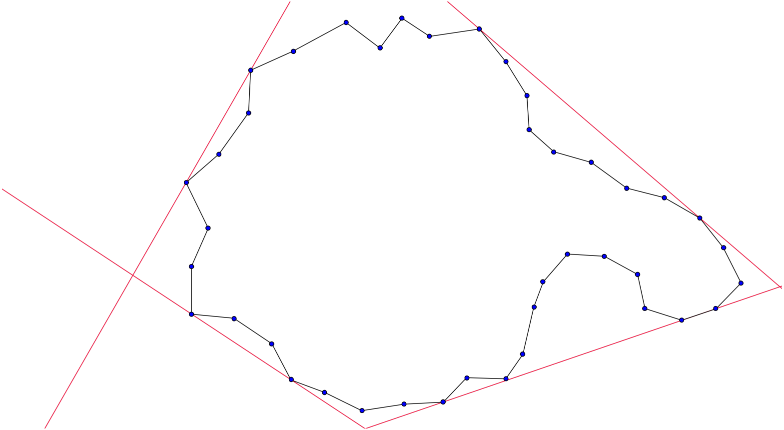

General 1-Cycles in $\mathbb{R}^n$
General 1-Cycles in $\mathbb{R}^{n-1}$
General 1-Cycles in $\mathbb{R}^2$
Convex 1-Cycles in $\mathbb{R}^2$
Uniform Convex 1-Cycles in $\mathbb{R}^2$
Regular Planar $k+1$ Polygon











Limiting Behavior

We know that if we can run this process in a nontrivial manner then the current manipulation of X must be nonconvex.

However, after ever nontrivial iteration of this process, we know that the volume of the polygon $V_\alpha(X)$ has increased. We also know that the area of any iteration is bounded (by isoperimetric inequality). Therefore, we know that there exists a limiting version of X to which we will no longer be able to apply the process. This implies convexity.

Preservation of α and $k+1$

Clearly, in the limiting version of X, call it $\tilde{X}$, the number of vertices will never change, thus $k+1$ is preserved

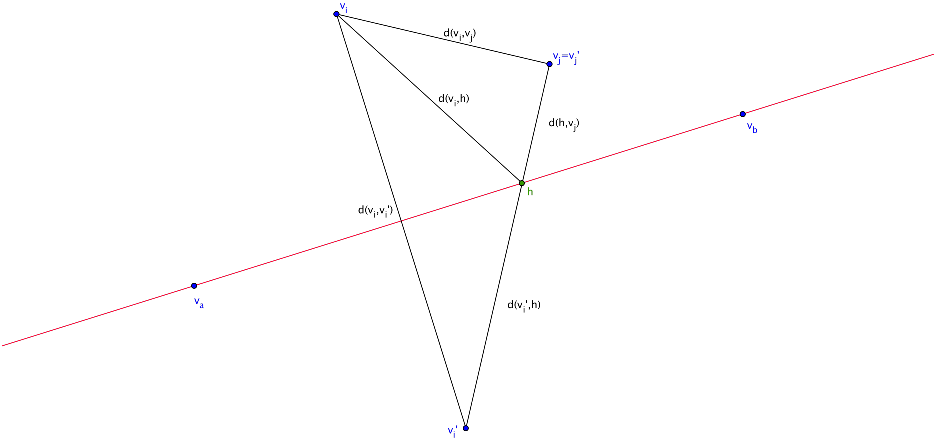
Moreover, pairwise distance between neighbors is maintained:

$$d(v_i, v_{i+1}) = d(v_i', v_{i+1}') = \cdots = d(\tilde{v}_i, \tilde{v}_{i+1})$$

Therefore, the birth value α is preserved.

Pairwise Distance Increase Justification

$$d(v_i', v_j') = d(v_i', v_j) = d(v_i', h) + d(h, v_j) = d(v_i, h) + d(h, v_j) \geq d(v_i, v_j)$$



Increase in γ

Because pairwise distances can only increase, the appearance of a 1-simplex in $V_\epsilon(X)$ can only be delayed, implying that the appearance of a 2-simplex in $V_\epsilon(X)$ can also only be delayed.

Because 2-simplices can only appear later, we know that complete coverings of the 1-cycle can only appear later as well, therefore the lifespan and γ must only have increased.

Therefore, we need only consider the set of convex 1-cycles when trying to maximize lifespan in the planar case.

Acknowledgments

Note that this slide was not included in the original presentation but I feel compelled to include it now.

I would first like to thank my graduate mentor, Glen Wilson, for his support and guidance as we continue to work on this problem and also Dr. Gene Fiorini and the DIMACS program for this opportunity.